Consolidated Week 7 Notes MAST30026 Sep 14, 2014

Lecture 1:

- **Example:** X a metric space.

$\mathcal{C}_b(X)$ = bounded continuous functions on X to $\mathbb{R}$ or $\mathbb{C}$.

$f \in \mathcal{C}_b(X)$, there exists k, $|f(x)| \leq k$ for all $x \in X$.

$\|f\|_{sup} = \sup_{x \in X} \{|f(x)|\}$

$(\mathcal{C}_b(X), \|\cdot\|_{sup})$ is a complete normed space, i.e. a Banach space.

Recall: $f_n \longrightarrow f$ in $\|\cdot\|_{sup} \iff f_n \to f$ uniformly $\Rightarrow f$ is continuous.

Check: $\|f_n - f\|_{sup} < \varepsilon \iff |f_n(x) - f(x)| < \varepsilon$ for all x.

$f_n \longrightarrow f$ pointwise $\not\Rightarrow$ f continuous.

$f(x) = x^n$ on $[0,1]$

But this jump discontinuity is annoying

So i use an "average norm"

$\|f-g\|_1 = \int_X |f-g| \quad X = \mathbb{R}^n$ and f,g continuous.

$\left(\ell^1 = \{(x_1, x_2, \dots) \mid \sum_{i=1} |x_i| < \infty\} \ \& \ \|x_i\|_1 = \sum_{i=1} |x_i| \right)$

This is a norm, but not complete.

***Def^n:** A step function. On $\mathbb{R}$, $[a,b]$ closed interval.

$$f(t) = \begin{cases} f(t) = 0, & t < a \text{ and } t > b \\ f(t) = c, & a \leq t \leq b \text{ and } c \neq 0. \end{cases}$$

If $a = b$, then it is a delta function.

On $\mathbb{R}^2$, $a_i \leq b_i$, $i = 1, 2, \dots$

$$f(x,y) = \begin{cases} f(x,y) = c, & a_1 \leq x \leq b_1 \\ & a_2 \leq y \leq b_2 \\ f(x,y) = 0, & \text{otherwise} \end{cases}$$

Note: Finite linear combinations are also called step functions

Note: Any continuous function is a limit of step functions. Very easy to integrate.

***Def^n:** An equivalence relation on functions is $f = g$ almost every where

if $\{x \mid f(x) = g(x)\}$ is a full set $\equiv \{x \mid f(x) \neq g(x)\}$ is a null set

(a) $f = f$ almost every where.

(b) If $f = g$ almost every where $\Longrightarrow$ $g = f$ almost everywhere

(c) If $f = g$ almost everywhere and $g = h$ almost everywhere $\Longrightarrow$ $f = h$ almost everywhere

$[f]$ equivalence class of f relative to almost everywhere.

- **Examples:** Look at all step functions which converge norm absolutely

$\mathcal{L}'(X)$ Lebesgue Integrable.

$\sum_{i=1}^{\infty} f_i$, f_i is a step function.

We require $\sum_{i=1}^{\infty} \|f_i\|_1 < \infty$

$\|f_i\|_1 = \int_X |f_i|$

But $\|\cdot\|_1$ is a seminorm since it fails the first norm axiom.

$(\|f\|_1 = 0 \not\Rightarrow x = 0)$

So $\mathcal{L}'(X) = \{ [f] \mid f = \sum_{i=1}^{\infty} f_i , \sum_{i=1}^{\infty} \|f_i\|_1 < \infty \}$

Now $\|\cdot\|_1$ is a true norm.

Now $\mathcal{L}^p(X) = \{ [f] \mid f = \sum_{i=1}^{\infty} f_i , \sum_{i=1}^{\infty} \|f_i\|_p < \infty \}$

$\|g\|_p = \left(\int |g(x)|^p \right)^{1/p}$

Banach space : $1 \leq p < \infty$.

***Def$^{\text{n}}$:** A **Hilbert space** is a vector space, V, with an inner product $\langle \cdot , \cdot \rangle$

If the metric is complete $(V, \langle \cdot , \cdot \rangle)$ is called a Hilbert space.

Note: Hilbert $\Longrightarrow$ Banach , $\|v\| = \sqrt{\langle v_1, v_2 \rangle}$

- **Examples:** $\mathcal{L}^p(X)$ is only a Hilbert space if $p = 2$.

$\mathcal{L}^2(X) = \{ [f] \mid f = \sum_{i=1}^{\infty} f_i , \sum_{i=1}^{\infty} \|f_i\|_2 < \infty \}$

$\langle f, g \rangle = \int f g$ (real case)

$= \int f \bar{g}$ (complex case).

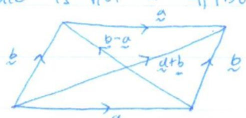
Then $\|f\|_2^2 = \langle f, f \rangle = \int f^2$ (real case)

$= \int f\bar{f} = \int |f|^2$ (complex case).

Lecture 2:

- **Example:** How do we tell if a Banach space is not a Hilbert space?

Use the parallelogram condition:



$$\|\underset{\sim}{a} + \underset{\sim}{b}\|^2 + \|\underset{\sim}{b} - \underset{\sim}{a}\|^2 = 2\|\underset{\sim}{a}\|^2 + 2\|\underset{\sim}{b}\|^2$$

Sum of square lengths of diagonals = Twice sum of squares of sides.

Valid in an inner product space.

Note: Necessary condition for $\|\cdot\|$ to come from $\langle \cdot , \cdot \rangle$

Note: Converse is true if scalars in $\mathbb{C}$.

$\mathcal{L}^p(X) = \{[f] \mid \int |f|^p < \infty\}$ $\quad 1 \leq p < \infty$.

This is not a Hilbert space since parallelogram condition fails ($p \neq 2$).

$\ell^p = \{x = (x_1, x_2, \dots) \mid \sum_{i=1}^{\infty} |x_i|^p < \infty\}$ is similarly Banach not Hilbert ($p \neq 2$).

*Def^n: Let $(V, \|\cdot\|)$ be a Banach space. A **Schauder basis** $\mathcal{B} = \{e_1, e_2, \dots\}$ is a countable set of vectors in V. Then every v in V can be written uniquely as a convergent series in $\mathcal{B}$.

$$v = \sum_{i=1}^{\infty} \lambda_i e_i$$

- Example: ℓ^p spaces, $1 \leq p < \infty$.

$e_1 = (1, 0, 0, \dots)$, $e_2 = (0, 1, 0, \dots)$, $e_3 = (0, 0, 1, 0, \dots)$

= "standard" Schauder basis $\mathcal{B}$.

$\underset{\sim}{x} = (x_1, x_2, \dots) = \sum_{i=1}^{\infty} x_i e_i$

Need to check uniqueness & convergence.

Recall that $\sum_{i=1}^{\infty} |x_i|^p < \infty$ by definition of ℓ^p spaces.

(Show $S_n = \sum_{i=1}^{n} x_i e_i$ is a Cauchy sequence or

$\|S_n - x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. Works since $\sum_{i=1}^{\infty} |x_i|^p < \infty$).

ie- $\|\sum_{i=1}^{\infty} x_i e_i\| < \varepsilon$ for $n \geq N$.

- Example: Non-example: $\ell^{\infty} = \{(x_1, x_2, \dots) \mid \sup |x_i| < \infty\}$

Try $\mathcal{B} = \{(1,0,0,\dots), (0,1,0,\dots), \dots\}$ as before?

$\underset{\sim}{x} = \sum_{i=1}^{\infty} x_i e_i$ but this series does not necessarily converge.

$\underset{\sim}{x} = (1,1,1,\dots)$. Series $\sum_{i=1}^{\infty} e_i$ does not converge.

No Schauder basis at all.

*Def^n: A set S in a Banach space $(V, \|\cdot\|)$ is called a **total set** if $\langle S \rangle$ = span of S = $\{$all finite linear combinations of vectors in $S\}$

= "subspace spanned by S"

is dense in V

So every v in V is a limit; $v = \lim_{n\to\infty} w_n$, $w_n \in \langle S \rangle$

- Proposition: A Schauder basis, $\mathcal{B}$, is a total set.

Remark: ℓ^p case

$\langle B \rangle$ = sequences $\underset{\sim}{x} = (x_1, x_2, \dots)$

where $x_i = 0$ for all but finitely many i.

*Def^n: A metric space is called **separable** if it has a countably dense set.

- Examples: (1) $\mathbb{R}$ is separable because $\mathbb{Q}$, the rationals, are countable & dense.

(2) ℓ^{∞} is not separable (Argument by contradiction, following Cantor).

- Proposition: If $(V, \|\cdot\|)$ is Banach & $\mathcal{B}$ is Schauder, then V is separable.

Lecture 3:

* **Def$^{?}$:** Characterizations of finite dimensional vector space.

  (a) Unit ball & unit sphere are compact

  (b) Linear transformations are continuous maps.

* **Def$^{?}$:** Characterizations of infinite dimensional vector space

  (a) Unit ball & unit sphere are never compact

  (b) Only bounded linear transformations are continuous maps.

* **HW:** Finite dimensional case. $(V, \|\cdot\|)$, $V \approx \mathbb{R}^n$ or $\mathbb{C}^n$

  show $B = \{x : \|x\| \leq 1\}$

  $S = \{x : \|x\| = 1\}$

  are closed & bounded.

* **HW:** Infinite dimensional case. If $(V, \|\cdot\|)$ is an infinite dimensional space, show that B and S are not compact.